THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION.

THE thirty-sixth regular meeting of the San Francisco Section was held at the University of California on Saturday, October 23. The chairman of the Section, Professor Blichfeldt, presided at the early part of the meeting; Professor Lehmer presided at the latter part. The attendance was twenty-two, including the following fifteen members of the Society:

Professor R. E. Allardice, Professor B. A. Bernstein, Professor H. F. Blichfeldt, Professor Florian Cajori, Professor M. W. Haskell, Professor Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Dr. F. R. Morris, Professor C. A. Noble, Professor T. M. Putnam, Dr. Pauline Sperry, Dr. S. E. Urner, Dr. A. R. Williams.

The following officers were elected for the year: chairman, Professor D. N. Lehmer; secretary, Professor B. A. Bernstein; programme committee, Professors H. F. Blichfeldt, W. A. Manning, B. A. Bernstein.

The dates of the next two meetings were provisionally fixed as April 9, 1921, and October 22, 1921.

The following papers were presented:

(1) Professor D. N. LEHMER: "On inverse ternary continued fractions."

(2) Professor M. W. HASKELL: "Curves autopolar with respect to two conics."

(3) Professor FLORIAN CAJORI: "Historical note on notations for ratio and proportion."

(4) Professor E. T. BELL: "The elliptic modular equation of the third order and the form $x^2 + 3y^2$."

(5) Professor E. T. BELL: "The reversion of class number relations and the total representation of an integer as a sum of square or triangular numbers."

(6) Professor E. T. BELL: "Class numbers and the form $xy + yz + zx$."

(7) Professor E. T. BELL: "Singly infinite class number relations."

(8) Professor E. T. BELL: "On recurrences for sums of divisors."

(9) Mr. H. W. BRINKMANN: "The group characteristics of the ternary linear fractional group and of various other groups."

(10) Professor H. F. BLICHFELDT: "Notes on geometry of numbers."

Mr. Brinkmann was introduced by Professor Manning. In the absence of the author the papers of Professor Bell were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. A ternary continued fraction is defined by means of its partial quotient pairs $(p_1, q_1; p_2, q_2; p_3, q_3; \cdots; p_n, q_n)$, the successive convergent sets (A_n, B_n, C_n) being solutions of the difference equation $u_n = q_n u_{n-1} + p_n u_{n-2} + u_{n-3}$, where the initial values for A_n, B_n, C_n are respectively, 1, 0, 0; 0, 1, 0; 0, 0, 1. (See the *Proceedings of the National Academy of Sciences*, volume 4, pages 360–364, December, 1918.)

It is of importance in the theory of cubic irrationalities to know what relations must exist among the partial quotient pairs in order that the two inverse ternary continued fractions $(p_1, q_1; p_2, q_2; \cdots; p_n, q_n)$, $(p_n, q_n; p_{n-1}, q_{n-1}; \cdots; p_1, q_1)$ should have the same characteristic cubic, the characteristic cubic being defined by the equation

$$\begin{vmatrix} A_{n-2} - \rho & B_{n-2} & C_{n-2} \\ A_{n-1} & B_{n-1} - \rho & C_{n-1} \\ A_n & B_n & C_n - \rho \end{vmatrix} = 0.$$

Professor Lehmer finds, besides the obvious case where the pairs read the same backward and forward, that the cubic is the same for both fractions if $p_i = \alpha t + \beta$, $q_i = \gamma t + \delta$, where $\alpha, \beta, \gamma, \delta$ are any fixed positive or negative integers or zero, while t is a variable parameter.

2. Professor Haskell described the configuration of four mutually autopolar conics, and gave a method for finding an unlimited number of higher plane curves autopolar with respect to two conics, when the latter are autopolar with respect to each other.

3. Professor Cajori notes that the colon was first used for ratio by the astronomer Vincent Wing in 1651. The forerunner

of Oughtred's notation $A \cdot B :: C \cdot D$ to signify $A : B = C : D$ is found in Billingsley's Euclid, 1570, where we find $9 \cdot 6 : 12 \cdot 8$. With Billingsley, the dot and colon were symbols of punctuation which were not yet limited to specific arithmetical use. Isaac Barrow wrote $A \cdot B + C \cdot D$ to signify the "compounding of ratios," i.e., $A : B$ times $C : D$. John Wallis opposed this practice. Sometimes, in the same equation, Barrow used $+$ to indicate the multiplication of ratios as well as the addition of terms.

4. In Professor Bell's first paper it is shown that the modular equation for the transformation of the third order in elliptic functions is implied by a theorem relating to even functions of two variables, the arguments of the functions being linear functions of the integers which represent an arbitrary integer in the form $x^2 + 3y^2$. The theorem admits of easy extension (for the appropriate quadratic forms), to the modular equation of the nth order, and takes particularly simple forms when n is a prime $4k + 1$, $12k + 7$ or the triple of a prime $12k + 1$. The paper will be published in the *Bulletin* of the Greek Mathematical Society.

5. In Professor Bell's second paper it is shown that the class number relations of Kronecker, Hermite and others may be reversed so as to give the class number for a negative determinant explicitly in terms of the total number of representations of certain integers each as a sum of square or triangular numbers, and further it is shown that each of a pair of such expressions (a relation and its inversion) implies the other. Recurrences for the computation of the new functions relating to total numbers of representations are given. These bear a striking resemblance to those for the class numbers.

6. In Professor Bell's third paper three general class number relations, each of which admits of specialization in an infinity of ways, are derived and their connection with the arithmetical form $xy + yz + zx$ considered. Each relation involves class numbers and an arbitrary even function of a single variable. On particularizing the function in various ways several special class number theorems are found at once, the simplest of which is: the total number of representations of n in the form

$xy + yz + zx$ for which $x, y, z > 0$, is $3[G(n) - 1]$ when and only when n is prime, $G(n)$ being the whole number of classes of binary quadratic forms for the determinant $-n$. The next simplest cases are famous class number relations due either to Kronecker or Hermite; the next are of the Liouville types, and thence onwards the special consequences are class number relations of kinds not hitherto stated. The paper will appear shortly in the *Tôhoku Mathematical Journal.*

7. A set of seventeen closely interrelated class number formulas of the kind described in Professor Bell's third paper is derived. In several ways the set is complete, no more results of the same general sort being implicit in the analysis. On specializing the arbitrary even functions involved in the most obvious ways, all of Kronecker's and Hermite's formulas drop out as the simplest cases, also some of those due to Liouville and Humbert. The method used was explained in the first part of a paper presented to the Society in December, 1918, which will appear in the *Transactions.*

8. A considerable section of Chapter X (volume I) of Dickson's History of the Theory of Numbers is devoted to divisor recurrences obtained from the expansions of elliptic functions. Professor Bell shows in this paper that any such recurrence is a very special case of a general relation between the divisors concerned, and specific application is made (in one of the illustrations) to well-known formulas of Halphen and Glaisher. A curious lacuna is observed: none of the formulas is sufficient for the calculation by recurrence alone of the $2r$th ($r = 0, 1, 2, \cdots$) powers of *all* the divisors, although the like may easily be done for the $(2r - 1)$th powers.

9. Mr. Brinkmann determines the group characteristics of the group of all ternary linear fractional substitutions of determinant unity whose coefficients are marks of any Galois field, and also the group characteristics of the group of all ternary linear fractional substitutions of non-vanishing determinant.

Further, the group characteristics of all primitive permutation groups of degree less than or equal to fifteen are determined, so far as they are not known already.

10. In Minkowski's development of geometry of numbers

(cf. this BULLETIN, volume 25 (1919), page 449) the following theorem is fundamental: if we designate by a Minkowski surface in R_n a finite surface in space of n dimensions, having as its chief characteristic a center of symmetry toward which it is nowhere convex (cf. l. c. for specific definition), then a Minkowski surface in R_n and of volume $\geqq 2^n$ will contain at least three distinct lattice points (i. e., points whose coordinates are integers) if its center is a lattice point. In order to extend the usefulness of the geometry of numbers, Professor Blichfeldt has amplified this theorem to read as follows: (1) a Minkowski surface in R_n of volume $\geqq 2^n k$ and whose center is a lattice point, must contain more than $k-1$ distinct pairs of lattice points in addition; (2) a Minkowski surface in R_n which contains k lattice points, its center being one, must have a volume $> (k-n)/n!$, if these k points do not all lie on a linear R_{n-1}. Some applications of this theorem were presented.

B. A. BERNSTEIN,
Secretary of the Section.

AN IMAGE IN FOUR-DIMENSIONAL LATTICE SPACE OF THE THEORY OF THE ELLIPTIC THETA FUNCTIONS.

BY PROFESSOR E. T. BELL.

(Read before the San Francisco Section of the American Mathematical Society June 18, 1920.)

1. In his memoir on "Rotations in space of four dimensions"* Professor Cole defined a system of four mutually orthogonal lineoids yzw, xzw, xyw, xyz (which we shall denote by X, Y, Z, W respectively) through a point O, the four lines and six planes determined by these, and with reference to this system found the transformations into itself of a sphere S with center at O. Henceforth we assume the radius of S to be $\sqrt{n}$, where n is an integer > 0. From this system we shall derive an image of the theory of the elliptic theta func-

* *Amer. Jour. of Math.*, vol. 12 (1890), p. 191.

tions by considering the reflexions of certain point configurations C, C', C'', C''', C^{iv} lying upon S with respect to O and the bisectors of the angles between X, Y, Z, W, a bisector of an angle between two lineoids being defined as a locus of points equidistant from the two. The space about O is latticed by four systems of lineoids parallel respectively to X, Y, Z, W, the successive lineoids in each system being at unit distances apart. We shall call this the unit lattice L. Any point all of whose coordinates are integers belongs to L; and conversely L contains only such points. Any integer > 0 being in several ways a sum of four integral squares, S always contains points of L, and these are symmetrical in pairs with respect to O.

Denote by L' the lattice containing all those points and only those whose coordinates are $(4a, 4b, 4c, 4d)$, where a, b, c, d take all integral values (including zero) from $-\infty$ to $+\infty$, and by $\alpha\beta\gamma\delta$ the lattice derived from L' by successive translations of L' through distances α, β, γ, δ parallel respectively to X, Y, Z, W, where α, β, γ, δ are integers $\geqq 0$, so that L' is 0000. There clearly are in all precisely 256 distinct lattices $\alpha\beta\gamma\delta$, each of which is contained in L, and these may be represented by symbols $\alpha'\beta'\gamma'\delta'$, where α', β', γ', δ' are the positive residues mod 4 of α, β, γ, δ. For brevity we assign current numbers to a special set of 64 contained in the 256. Only the $\alpha'\beta'\gamma'\delta'$ wherein α', β' are both even or both odd are required in the sequel, and likewise for γ', δ'. The requisite half-symbols $\alpha'\beta'$, $\gamma'\delta'$ are therefore 00, 02, 11, 13, 20, 22, 31, 33. Write

$$20, 22, 02, 00 \equiv 2, 4, 6, 8,$$

$$11, 13, 31, 33 \equiv 1, 3, 5, 7,$$

respectively. In this notation the lattice 0222 is 64; the current number of 3320 is 72; that of 1311 is 31; 3100 is 58; 0231 is denoted by 65, etc. To signify that all the points of a certain configuration C belong to one of these lattices, say to ij, we give C the corresponding double suffix, C_{ij}. The theory of the theta functions is formally equivalent to the symmetries of certain C_{ij} lying upon S. By the formal equivalence of A, B we mean that each implies the other.

2. Consider first any point P lying within (not on any of X, Y, Z, W) any one T of the sixteen right tetraedral angles into which space is partitioned by X, Y, Z, W. Bisect the

angles which any one of X, Y, Z, W, say X, makes with the remaining three, and denote by Y', Z', W' those parts of the bisectors which lie within T, and by U', V', U'', V'', U''', V''' the like parts of the bisectors of the angles between the three pairs of opposite pairs X, Y and Z, W; X, Z and Y, W; X, W and Y, Z, viz., U' is the bisector for X, Y; V' for Y, Z, etc. (We have chosen the internal bisectors with respect to the angles of T.) Reflect P in Y', reflect the image in Z', and reflect this image in W', getting finally the point P_1. In whatever order the three successive reflexions are performed, it is clear that the same P_1 is reached. From P_1 in the same way derive P_2, from P_2 similarly P_3, and from P_3 in the same way P_4. Then $P_4 \equiv P$. Second, if P lies on at least one of X, Y, Z, W, we avoid ambiguities (of sign) by requiring the reflexions to be performed so that the signs of the coordinates of P are unchanged; e.g., the signs being $(++--)$ are to be the same before and after reflexion. Reflect P in O, getting P_0; join the centroid Π of P_0, P_1, P_2, P_3 to O, and produce $O\Pi$ through Π to cut S in P', which point we shall call the mate of P. Reflect P' in U', reflect the image in V', getting P'', called the first skew mate of P. Similarly from P' and U'', V'' get P''', and from U''' and V''', get P^{iv}, the second and third skew mates of P. Note that $P', \cdots, P^{\text{iv}}$ are significant only with respect to the particular T in which P lies. Taking the mates of all points in any configuration C we get its mate C', and similarly for the first, second and third skew mates C'', C''', C^{iv}.

We shall be concerned with two kinds of symmetry about O of C_{ij} and their mates. If A, B are any configurations such that each may be brought into coincidence with the other by reflexions in O of some (or all) of its points, A, B are called images of each other. All those points of any configuration C which are such that no one of them is the reflexion in O of another, form a configuration called the residue of C; and any configurations A_1, B_1 are said to be skew images of each other when their residues coincide.

Finally we need also the idea of lattice configurations with multiple points. Let each point of C_1, C_2, $\cdots$, C_r belong to the unit lattice. By $C_1 + C_2 + \cdots + C_r, \equiv C$, we mean the configuration which consists of all the points of C_1, C_2, $\cdots$, C_r. A point occurring in precisely s of the C_i is multiple of order s in C; and two coincident configurations are identical when

and only when points occupying the same position in both are of equal multiplicities. In particular if A, B are images or skew images of each other, the points in any pair which are regarded as reflexions in O of one another must be of the same multiplicity. To indicate that each point in C is of multiplicity s, we write sC.

We can now give the image of the single theta functions. The geometrical theorems will first be stated, their theta equivalents then pointed out, and the means for passing from one to the other briefly indicated. The geometry can be derived simply from first principles. It is shorter, however, to proceed as in § 4. The eleven images can be compressed into one relating to S and the unit lattice, but the statement is complicated. The eleven exhibit a manifold symmetry recalling that of crystals, which becomes evident when the C_{ij} and their mates and skew mates C_{ij}', C_{ij}'', etc., are written with the suffixes in full, thus, C_{0011}, C_{3320}'. In all that follows the C_{ij}, and therefore also their mates and skew mates, are configurations of points lying on the S of radius $\sqrt{n}$ defined in § 1.

3. The first two theorems concern the case $n \equiv 0 \bmod 4$, and the configurations

$$C_0 \equiv C_{22} + C_{26} + C_{62} + C_{66}, \quad C_2 \equiv C_{24} + C_{28} + C_{64} + C_{68},$$

$$C_4 \equiv C_{44} + C_{48} + C_{84} + C_{88}, \quad C_6 \equiv C_{42} + C_{46} + C_{82} + C_{86};$$

$$C_1 \equiv C_{11} + C_{17} + C_{71} + C_{77}, \quad C_3 \equiv C_{13} + C_{15} + C_{73} + C_{75},$$

$$C_5 \equiv C_{33} + C_{35} + C_{55} + C_{53}, \quad C_7 \equiv C_{31} + C_{37} + C_{51} + C_{57}.$$

THEOREM I. *Each of $C_0 + C_4$, $C_1 + C_5$ is the image of its mate, and each of $C_2 + C_6$, $C_3 + C_7$ is the image of the mate of the other.*

THEOREM II. *The configurations in each of the following pairs are images of each other:*

$$C_0 + C_2 + C_4' + C_6' + C_1 + C_3 + C_5' + C_7',$$

$$C_0' + C_2' + C_4 + C_6 + C_1' + C_3' + C_5 + C_7;$$

$$C_0 + C_2 + C_2' + C_4' + C_1' + C_5 + C_7 + C_7',$$

$$C_0' + C_4 + C_6 + C_6' + C_1 + C_3 + C_3' + C_5'.$$

The next two are for $n \equiv 2 \bmod 4$, and the configurations

$$B_1 \equiv C_{12} + C_{16} + C_{72} + C_{76}, \quad B_3 \equiv C_{34} + C_{38} + C_{54} + C_{58},$$

$$B_5 \equiv C_{14} + C_{18} + C_{74} + C_{78}, \quad B_7 \equiv C_{32} + C_{36} + C_{52} + C_{56}.$$

THEOREM III. *$B_1 + B_3$ is the image of its second skew mate, and $B_5 + B_7$ is the image of its mate.*

THEOREM IV. *$B_1 + B_3'''$ is the image of $B_1''' + B_3$, and $B_5' + B_7$ is the image of $B_5 + B_7'$.*

The next is also for $n \equiv 2 \bmod 4$, and the configurations

$$D_1 \equiv C_{16} + C_{18} + C_{36} + C_{38} + C_{52} + C_{54} + C_{72} + C_{74},$$

$$D_2 \equiv C_{12} + C_{14} + C_{32} + C_{34} + C_{56} + C_{58} + C_{76} + C_{78}.$$

THEOREM V. *The following are skew images of each other:*

$$2D_1 + D_1' + D_2'' + D_2''' + D_2^{\text{iv}},$$

$$2D_2 + D_2' + D_1'' + D_1''' + D_1^{\text{iv}}.$$

4. The foregoing theorems imply the theory of the theta functions. For, if f, g are single-valued functions of four variables existing when each variable takes integral values $\gtrless 0$, such that

$$f(x, y, z, w) = \quad f(-x, -y, -z, -w),$$

$$g(x, y, z, w) = -g(-x, -y, -z, -w),$$

and otherwise are wholly arbitrary, we may express that A, B are images, that A, B are skew images, by

$$\Sigma f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \Sigma f(\beta_1, \beta_2, \beta_3, \beta_4),$$

$$\Sigma g(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \Sigma g(\beta_1, \beta_2, \beta_3, \beta_4)$$

respectively, the Σ's extending to all points $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of A, and $(\beta_1, \beta_2, \beta_3, \beta_4)$ of B. On remarking that if $P \equiv (x_1, x_2, x_3, x_4)$, then P', P'', P''', $P^{\text{iv}} = (x_1', x_2', x_3', x_4')$, (x_2', x_1', x_4', x_3'), (x_3', x_4', x_1', x_2'), (x_4', x_3', x_2', x_1'), where $x_i' \equiv s - x_i$ and $2s = x_1 + x_2 + x_3 + x_4$, we may easily verify that the five theorems are equivalent to the following analytical restatements of them. The m_i denote odd integers $\gtrless 0$, the l_i even integers $\gtrless 0$, representing $n > 0$ in the forms

$$\sum_{i=1}^{4} m_i^2, \qquad \sum_{i=1}^{4} l_i^2, \qquad m_1^2 + m_2^2 + l_3^2 + l_4^2,$$

and summations are with respect to all l_i, m_i for a given n. The rest of the notation is:

$$2\lambda_1 = l_1 + l_2, \quad 2\lambda_3 = l_3 + l_4, \quad 2\mu_1 = m_1 + m_2, \quad 2\mu_3 = m_3 + m_4,$$
$$\lambda = \lambda_1 + \lambda_3, \quad \mu = \mu_1 + \mu_3, \quad \nu = \mu_1 + \lambda_3, \quad 2\alpha = m_1 - 1 + l_3,$$
$$l_i' = \lambda - l_i, \quad m_i' = \mu - m_i, \quad l_i'' = \nu - l_i, \quad m_i'' = \nu - m_i.$$

Corresponding to theorems (I)–(V) we now have the following:

(I′) $$\Sigma[(-1)^{\mu} + 1]f(m_1, m_2, m_3, m_4) = \Sigma[(-1)^{\mu} + 1]f(m_1', m_2', m_3', m_4'),$$
$$\Sigma[(-1)^{\mu} - 1]f(m_1, m_2, m_3, m_4) = \Sigma[(-1)^{\lambda} - 1]f(l_1', l_2', l_3', l_4'),$$
$$\Sigma[(-1)^{\lambda} + 1]f(l_1, l_2, l_3, l_4) = \Sigma[(-1)^{\lambda} + 1]f(l_1', l_2', l_3', l_4'),$$
$$\Sigma[(-1)^{\lambda} - 1]f(l_1, l_2, l_3, l_4) = \Sigma[(-1)^{\mu} - 1]f(m_1', m_2', m_3', m_4').$$

(II′) $$\Sigma[(-1)^{\mu_1}f(m_1, m_2, m_3, m_4) + (-1)^{\lambda_1}f(l_1, l_2, l_3, l_4)] = \Sigma[(-1)^{\mu_1}f(m_1', m_2', m_3', m_4') + (-1)^{\lambda_1}f(l_1', l_2', l_3', l_4')],$$
$$\Sigma[(-1)^{\mu_1}f(m_1, m_2, m_3, m_4) - (-1)^{\lambda_1}f(l_1, l_2, l_3, l_4)] = \Sigma[(-1)^{\mu_2}f(m_1', m_2', m_3', m_4') - (-1)^{\lambda_2}f(l_1', l_2', l_3', l_4')].$$

(III′) $$\Sigma[(-1)^{\nu} + 1]f(m_1, m_2, l_3, l_4) = \Sigma[(-1)^{\nu} + 1]f(l_3'', l_4'', m_1'', m_2''),$$
$$\Sigma[(-1)^{\nu} - 1]f(m_1, m_2, l_3, l_4) = \Sigma[(-1)^{\nu} - 1]f(m_1'', m_2'', l_3'', l_4'').$$

(IV′) $$\Sigma[(-)^{\mu_1} + (-1)^{\lambda_3}]f(m_1, m_2, l_3, l_4) = \Sigma[(-1)^{\mu_1} + (-1)^{\lambda_3}]f(l_3'', l_4'', m_1'', m_2''),$$
$$\Sigma[(-1)^{\mu_1} - (-1)^{\lambda_3}]f(m_1, m_2, l_3, l_4) = \Sigma[(-1)^{\mu_1} - (-1)^{\lambda_3}]f(m_1'', m_2'', l_3'', l_4'').$$

(V′) $$2\Sigma(-1)^{\alpha}g(m_1, m_2, l_3, l_4) = \Sigma(-1)^{\alpha}[g(l_4', l_3', m_2', m_1') + g(l_3', l_4', m_1', m_2') + g(m_2', m_1', l_4', l_3') - g(m_1', m_2', l_3', l_4')].$$

To pass to the theta functions, replace $f(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $g(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ by the cos, sin respectively of $(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4)$, where x_1, x_2, x_3, x_4 are parameters. In this form the trigonometric sums involving unaccented letters in I′–IV′, and the left of the identity in V′, are readily seen to be the coefficients of q^{4n} or q^{4n+2} in products of the form

$$\vartheta_\alpha(x_1, q^4)\vartheta_\beta(x_2, q^4)\vartheta_\gamma(x_3, q^4)\vartheta_\delta(x_4, q^4),$$

where $\alpha, \beta, \gamma, \delta$ are certain of the numbers 0, 1, 2, 3. Again, the sums involving accented letters in the trigonometric forms of I′–IV′ and the right of V′ are likewise seen to be the coefficients of the same powers of q collected from four such theta products, taken with appropriate signs, in which the variables are $x_i' \equiv s - x_i$, where $2s = x_1 + x_2 + x_3 + x_4$. In this way we find I′–V′ in their trigonometric cases to imply eleven theta identities, which, as they are easily accessible in H. J. S. Smith's second paper on the multiplication formula of four theta functions (Papers, volume 2, page 279), we need not transcribe. It will be sufficient to state the particular formulas of Smith which the trigonometric forms of I′–V′ thus imply. The results in I′ give Smith's (i) $\pm$ (ii),(iii) $\pm$ (iv), and therefore (i) $-$ (iv) in his set A. Similarly his B, C are implied by our II′, III′; his (ix), (xi) of D by our IV′, and his (x) by our V′. From these eleven independent theta formulas, the theory of the theta and elliptic functions, as is well known, follows readily; in fact Smith so derives the theory in his "Memoir on the theta and omega functions" (Papers, volume 2, page 415). The analogous derivation by Jacobi in a famous memoir (Werke, volume 1, page 499) differs only in details. His set (A) is the equivalent of Smith's A–D. We have followed Smith's development rather than Jacobi's because it is the more symmetrical. Jacobi's gives another geometrical image of the theory, and Kronecker's well known exposition of Jacobi's methods yet a third, in which the simpler regular solids inscribed in S play an interesting part.

Now conversely I′–V′, and therefore I–V are implied by the eleven theta formulas of Smith. This follows immediately from the method of paraphrase* outlined in this BULLETIN, volume 26, page 220, § 13. Hence I–V imply, and are implied by, the theory of the elliptic theta functions; viz., the two are formally equivalent.

* The proofs of the method are given in "Arithmetical paraphrases," Part I, to appear shortly in the *Transactions*

5. There is an analogous image for the theory of the theta functions of r variables. In it the lattice space is of $2r(r+1)$ dimensions, and the appropriate configurations lie on a system of r four-dimensional spheres immersed in the higher space. For $r > 1$ the image is not expressible in terms of reflexions alone.

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NOTE ON THE MEDIAN OF A SET OF NUMBERS.

BY PROFESSOR DUNHAM JACKSON.

(Read before the American Mathematical Society September 7, 1920.)

LET $a_1, a_2, \cdots, a_n$ be a set of real numbers, which may or may not be all distinct. Let

$$S_2(x) = \sum_{i=1}^{n} (x - a_i)^2.$$

The value of x which reduces $S_2(x)$ to a minimum is the arithmetical mean of the numbers $a_1, \cdots, a_n$. If the condition that $S_2(x)$ be a minimum is replaced by the condition that

$$S_1(x) = \sum_{i=1}^{n} |x - a_i|$$

be reduced to a minimum, the median of the a's is obtained. It is uniquely defined whenever n is odd; if the numbers a_i are arranged in order of magnitude, so that

$$a_1 \leqq a_2 \leqq \cdots \leqq a_n,$$

and if $n = 2k - 1$, the median is simply a_k, the middle one of the a's. The median is uniquely defined also when n is even, $n = 2k$, if it happens that $a_k = a_{k+1}$, being then equal to this common value. Otherwise, the definition is satisfied by any number x belonging to the interval

$$a_k \leqq x \leqq a_{k+1},$$

and the median is to this extent indeterminate.

The purpose of the following paragraphs is to show that

for each value of $p > 1$ there is a definite number $x = x_p$ which minimizes the sum

$$S_p(x) = \sum_{i=1}^{n} |x - a_i|^p,$$

and that x_p approaches a definite limit X as p approaches 1. The value of X coincides with the median as already defined, in the cases where that definition is determinate; and when $n = 2k$ and $a_k \neq a_{k+1}$, X is a definite number between a_k and a_{k+1}. It serves thus to supplement the former definition, on the theoretical side at any rate.

It is not unlikely that the same average has been discussed, as a curiosity at least, by writers on statistics. The accurate proof of the statements involved, however, seems to be an exercise in pure mathematics. It will have to be admitted that the number does not lend itself readily to direct computation, except in the simplest cases.

In the first place, for any particular value of $p > 1$, $S_p(x)$ is a continuous function of x which is always positive or zero, and which becomes infinite as x goes out to infinity in either direction. Consequently it attains a minimum for some one value of x at least.

Furthermore, each term $|x - a_i|^p$ has a continuous derivative for all values of x, the value $x = a_i$ not excepted. Hence $S_p(x)$ also has a continuous derivative everywhere, and this derivative must vanish at any point where a minimum is reached. But the derivative of $|x - a_i|^p$ always increases when x increases; hence the same thing is true of the derivative of $S_p(x)$, and $S_p'(x)$ can vanish only once. That is, the minimizing value x_p is indeed uniquely determined, and is moreover characterized by the vanishing of $S_p'(x)$.

To be more precise with regard to the formulas involved,

$$\frac{d}{dx}|x - a_i|^p = \frac{d}{dx}(x - a_i)^p = \quad p(x - a_i)^{p-1}, \qquad x > a_i;$$

$$\frac{d}{dx}|x - a_i|^p = \frac{d}{dx}(a_i - x)^p = -\,p(a_i - x)^{p-1}, \qquad x < a_i;$$

while for $x = a_i$ the derivative is correctly represented by either formula, being equal to zero. If i is any one of the indices $1, 2, \cdots, n - 1$, and x any number in the interval

$a_i \leqq x \leqq a_{i+1}$,

$$S_p(x) = (x - a_1)^p + \cdots + (x - a_i)^p + (a_{i+1} - x)^p + \cdots + (a_n - x)^p,$$

$$(1) \quad \frac{1}{p} S_p'(x) = (x - a_1)^{p-1} + \cdots + (x - a_i)^{p-1} - (a_{i+1} - x)^{p-1} - \cdots - (a_n - x)^{p-1}.$$

Now suppose n is odd, $n = 2k - 1$. Let ϵ be any positive quantity, and let r be an index such that

$$a_r < a_k + \epsilon \leqq a_{r+1}.$$

Then $r \geqq k$. By the formula just written down,

$$\frac{1}{p} S_p'(a_k + \epsilon) = (a_k + \epsilon - a_1)^{p-1} + \cdots + (a_k + \epsilon - a_r)^{p-1} - (a_{r+1} - a_k - \epsilon)^{p-1} - \cdots - (a_n - a_k - \epsilon)^{p-1}.$$

When p approaches 1, each of the first k terms on the right approaches 1 as a limit. Any one of the remaining $k - 1$ terms, sign included, has the limit 1, 0, or -1, as the case may be. At any rate,

$$\lim_{p=1} S_p'(a_k + \epsilon) > 0.$$

There exists a positive δ_1 such that

$$S_p'(a_k + \epsilon) > 0$$

whenever $1 < p \leqq 1 + \delta_1$.

Similarly, there is a positive δ_2 such that

$$S_p'(a_k - \epsilon) < 0$$

if $1 < p \leqq 1 + \delta_2$. If δ is the smaller of δ_1, δ_2, and $1 < p \leqq 1 + \delta$, it is certain that $S_p'(x)$ vanishes between $a_k - \epsilon$ and $a_k + \epsilon$. But the point where $S_p'(x)$ vanishes is the point x_p; consequently

$$a_k - \epsilon < x_p < a_k + \epsilon$$

for $1 < p \leqq 1 + \delta$, that is,

$$\lim_{p=1} x_p = a_k.$$

Analogous reasoning shows that when $n = 2k$ and $a_k = a_{k+1}$,

$$\lim_{p=1} x_p = a_k = a_{k+1}.$$

It remains to consider the case of principal interest, the case that $n = 2k$ and $a_k < a_{k+1}$. By reasoning similar to that already employed, it is seen that $\lim_{p=1} S_p'(a_k) < 0$. For at least k terms in the expression for $S_p'(a_k)$ approach the limit -1, the term $(x - a_k)^{p-1}$ is zero, and not more than $k - 1$ terms are positive. In the same way, $\lim_{p=1} S_p'(a_{k+1}) > 0$, and therefore

$$a_k < x_p < a_{k+1}$$

when p is sufficiently near to 1.

When $x > a_i$, $(x - a_i)^{p-1}$ can be represented by the series

$$\begin{aligned}(x - a_i)^{p-1} &= e^{(p-1)\log(x-a_i)} \\ &= 1 + (p-1)\log(x - a_i) + \tfrac{1}{2}(p-1)^2 \log^2(x - a_i) + \cdots \\ &= 1 + (p-1)\log(x - a_i) + (p-1)^2\varphi_i(x, p),\end{aligned}$$

where $\varphi_i(x, p)$ is a function which remains finite if x is fixed and p approaches 1. Similarly,

$$(a_i - x)^{p-1} = 1 + (p-1)\log(a_i - x) + (p-1)^2\varphi_i(x, p)$$

when $x < a_i$. The work so far has been based on the use of the first term as an approximation to the value of the series; it will be convenient now to make use of the explicit form of the first two terms.

If the expressions just indicated are substituted in the right-hand member of (1), for a value of x between a_k and a_{k+1}, there will be k terms each equal to $+1$ and k terms each equal to -1, which will cancel, and a relation will be obtained which can be written in the form

$$\begin{aligned}\frac{1}{p(p-1)} S_p'(x) &= \log(x - a_1) + \cdots + \log(x - a_k) \\ &\quad - \log(a_{k+1} - x) - \cdots - \log(a_n - x) \\ &\quad + (p-1)\varphi(x, p) \\ &= \log \frac{(x - a_1)\cdots(x - a_k)}{(a_{k+1} - x)\cdots(a_n - x)} + (p-1)\varphi(x, p),\end{aligned}$$

where $\varphi(x, p)$ is a function which remains finite for fixed x as p approaches 1.

The fraction on the right in the last relation is equal to 0 for $x = a_k$; it increases steadily when x increases from a_k

toward a_{k+1}, each factor of the numerator increasing while each factor of the denominator decreases; and it becomes infinite as x approaches a_{k+1}. Hence there is just one intermediate point, $x = X$, at which the fraction is equal to 1 and its logarithm to 0.

For $x = X - \epsilon$, where ϵ is an arbitrarily small positive quantity, the logarithm has a negative value, independent of p. Hence $S_p'(X - \epsilon)$ is negative if $p - 1$ is sufficiently small. Similarly, $S_p'(X + \epsilon)$ is positive for small values of $p - 1$. This means that x_p is between $X - \epsilon$ and $X + \epsilon$ if p is sufficiently near to 1, that is,

$$\lim_{p=1} x_p = X.$$

So the original assertion is proved.

The number X is characterized by the equation

$$(X - a_1) \cdots (X - a_k) = (a_{k+1} - X) \cdots (a_n - X).$$

If $n = 2$,

$$X = \frac{a_1 + a_2}{2}.$$

If $n = 4$,

$$X = \frac{a_4 a_3 - a_2 a_1}{(a_4 + a_3) - (a_2 + a_1)}.$$

When $n > 4$, the explicit determination of X will involve the solution of an algebraic equation of more or less high degree.

The discussion has been concerned primarily with values of p not much greater than 1. The proof of the existence and uniqueness of x_p, however, applies equally well when p is arbitrarily large. It is readily seen that

$$\lim_{p=\infty} x_p = Y = \tfrac{1}{2}(a_1 + a_n),$$

the intermediate a's having no effect on the result. For if ϵ is fixed and p is very large, the single term $(x - a_1)^p$ will be greater than the whole sum $S_p(Y)$ when $x \geqq Y + \epsilon$, with a corresponding inequality on the other side, so that x_p must be between $Y - \epsilon$ and $Y + \epsilon$.

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NOTE ON CLOSURE OF ORTHOGONAL SETS.

BY PROFESSOR O. D. KELLOGG.

(Read before the American Mathematical Society April 24, 1920.)

1. *Introduction.*

THE present note has to do with the set $[\varphi_i(x)]$ $(i = 0, 1, 2, \cdots)$ of solutions of a differential equation

$$\frac{d}{dx}(k\varphi') + (\lambda g - l)\varphi = 0, \tag{1}$$

in which k, g, and l are continuous, and $k > 0$ $(a \leqq x \leqq b)$, $g > 0$ $(a < x < b)$, the solutions satisfying a pair of homogeneous linear self-adjoint boundary conditions

$$\begin{aligned} U_1(\varphi) &= a_1\varphi(a) + a_2\varphi'(a) - a_3\varphi(b) - a_4\varphi'(b) = 0, \\ U_2(\varphi) &= b_1\varphi(a) + b_2\varphi'(a) - b_3\varphi(b) - b_4\varphi'(b) = 0, \end{aligned} \tag{2}$$

the condition for self-adjointness being

$$k(a)(a_3b_4 - a_4b_3) = k(b)(a_1b_2 - a_2b_1). \tag{3}$$

Such a set is called *closed* with respect to functions of a given class, provided there is no function of the class orthogonal to all the functions of the set, i.e., no function f such that $\int_a^b f\varphi_i g\,dx = 0$, $(i = 0, 1, 2, \cdots)$. Closure of a set is evidently implied whenever it is known that a function of a given class is uniquely determined by its generalized Fourier constants, and in this way a large number of closure theorems are at hand. Stekloff* has proved that for the case of the Sturm-Liouville boundary conditions, in which $a_3 = a_4 = b_1 = b_2 = 0$, the set of solutions is closed with respect to the class of functions that are integrable with integrable squares. Hilbert,† using boundary conditions in part special cases of (2) above, and in part going beyond, in that singularities of the differential

* *Annales de la Faculté des Sciences de Toulouse*, 2d ser., vol. 3 (1901); *Mémoires de l'Académie Impériale des Sciences de St.-Petersbourg*, vol. 30, No. 4 (1911).

† *Göttinger Nachrichten*, 1904, p. 222.

equation at the end points of the interval (a, b) are considered, shows that the sets studied are closed with respect to the class of continuous functions. In the following, I wish first to call attention to an easily derived identity (7) from which it may be inferred immediately that *if the set of solutions of the differential system* (1) *and* (2) *is closed with respect to the class C of continuous functions not zero, it is then also closed with respect to the class S of summable* functions not null functions.* A second paragraph will indicate an application of the method of successive approximations from which the existence of the set of solutions may be inferred, as well as its closure with respect to functions of class S.

2. *The Relative Closure Identity.*

Let $f(x)$ be a function of class S, and let μ be a value of λ for which the system (1) and (2) has no solution except 0. Then the differential equation

$$\frac{d}{dx}(kw') + (\mu g - l)w = fg \tag{4}$$

has a solution in the following sense: there exists a function $w(x)$, continuous, together with its first derivative, which satisfies the boundary conditions (2), and which satisfies (4) except at most at points of a set of measure 0. To see this, take two solutions, $v_1(x)$ and $v_2(x)$, of the homogeneous equation obtained from (4) by replacing the right-hand member by 0; these may be chosen so that $k(x)(v_1v_2' - v_1'v_2) \equiv 1$. Then

$$w_0(x) = \int_a^x \begin{vmatrix} v_1(\xi), & v_2(\xi) \\ v_1(x), & v_2(x) \end{vmatrix} f(\xi)g(\xi)d\xi \tag{5}$$

is continuous, since the indefinite integral of a summable function is continuous. The same is true of its derivative. Finally, kw', since the indefinite integral of a summable function has a derivative equal to the integrand, except at most at points of a set of zero measure (Lebesgue, l. c., page 124–125), has a derivative except at points of a null set. It is immediately seen that in this sense $w_0(x)$ is a particular solution of (4).

* In the sense of Lebesgue. See Leçons sur l'Intégration et la Recherche des Fonctions primitives, p. 115.

From $w_0(x)$ one forms the solution which satisfies (2):

$$(6)\quad w(x) = \begin{vmatrix} w_0(x) & v_1(x) & v_2(x) \\ U_1(w_0) & U_1(v_1) & U_1(v_2) \\ U_2(w_0) & U_2(v_1) & U_2(v_2) \end{vmatrix} \div \begin{vmatrix} U_1(v_1) & U_1(v_2) \\ U_2(v_1) & U_2(v_2) \end{vmatrix},$$

where the denominator determinant does not vanish, since μ was a parameter value for which the homogeneous problem was not possible. Let now $\varphi_i(x)$ be any one of the set of solutions of (1) and (2), and let λ_i be the corresponding value of λ. From (1) and (4) by a familiar process, we get

$$\frac{d}{dx}k(\varphi_i w' - \varphi_i' w) + (\mu - \lambda_i)w\varphi_i g = f\varphi_i g.$$

This equation, failing at most on a set of measure 0, becomes an identity upon integration between the limits a and b. Since φ_i and w satisfy (2), which are self-adjoint, and (3), the integrated terms disappear, and we are led to

$$(7)\qquad (\mu - \lambda_i)\int_a^b w\varphi_i g dx = \int_a^b f\varphi_i g dx,$$

the identity referred to in the introduction. Since f is not identically zero, w cannot be, by (4), and so is of class C. The identity shows that w is orthogonal to every function φ_i to which f is orthogonal and the theorem stated follows.

3. *Existence and Closure of the Set* $[\varphi_i(x)]$.

We start with the functions f and w above, without, however, supposing anything about their orthogonality to functions of the set $[\varphi_i(x)]$. We define a sequence $w_1, w_2, w_3, \cdots$ as follows:

$$w_{i0} = \int_a^x \begin{vmatrix} v_1(\xi), & v_2(\xi) \\ v_1(x), & v_2(x) \end{vmatrix} \overline{w}_{i-1}(\xi)g(\xi)d\xi,$$

$$(8)\quad w_i = \begin{vmatrix} w_{i0}(x) & v_1(x) & v_2(x) \\ U_1(w_{i0}) & U_1(v_1) & U_1(v_2) \\ U_2(w_{i0}) & U_2(v_1) & U_2(v_2) \end{vmatrix} \div \begin{vmatrix} U_1(v_1) & U_1(v_2) \\ U_2(v_1) & U_2(v_2) \end{vmatrix},$$

$$\overline{w}_i = w_i \Big/ \sqrt{\int_a^b w_i^2 g dx},$$

$$w_1 = w.$$

These functions satisfy at all points the differential equations

$$\frac{d}{dx}(kw_i') + (\mu g - l)w_i = \overline{w}_{i-1}g. \tag{9}$$

We proceed to consider the existence and properties of limit functions of the sequence $[w_i]$. We first note that w_i and w_i' are bounded by a number B independent of i. For, if M is greater than the maxima of $|v_1|$, $|v_1'|$, $|v_2|$, $|v_2'|$, and g, we have by Schwarz's inequality applied to (8_1), $|w_i|$ and $|w_i'| < 2M\sqrt{M(b-a)}$, so that a bound, B, for $|w_i|$ and $|w_i'|$ is easily obtainable.

Next, writing the equation (9) and the same equation with $i+1$ replacing i, multiplying these equations respectively by $\overline{w}_{i+1}$ and $\overline{w}_i$, subtracting, and integrating from a to b, we find $\int_a^b w_{i+1}\overline{w}_{i-1}gdx = \int_a^b w_i\overline{w}_i gdx$. If in this equation we replace the value of the normed functions by their values in terms of the un-normed functions, we find

$$\int_a^b w_{i+1}w_{i-1}gdx = \sqrt{\int_a^b w_i^2 gdx \cdot \int_a^b w_{i-1}^2 gdx}. \tag{10}$$

From this we infer two things. First, by Schwarz's inequality, that $\int_a^b w_i^2 gdx \leqq \int_a^b w_{i+1}^2 gdx$, and hence, that the bounded numbers $\int_a^b w_i^2 gdx$ approach a limit c^2, and secondly, that the integral $\int_a^b w_i w_{i+2} gdx$ also approaches c^2.

We now know that because of the boundedness of the derivatives of w_i, there exists a sub-sequence taken from the w_{2i} which approaches a limit W_0.* This limit is approached uniformly, and is continuous. If we denote by $[w_{2i'}]$ the subsequence approaching W_0 uniformly, then (8) shows that $w_{2i'+1}$ also approaches a limit W_1, and that this function satisfies the differential equation

$$\frac{d}{dx}(kW_1') + (\mu g - l)W_1 = W_0 g/c. \tag{11}$$

*Cf. Osgood, *Annals of Mathematics*, vol. 14 (1913), p. 182.

Similarly $w_{2\nu+2}$ approaches a limit, W_2, say. But this limit cannot differ from W_0, since

$$\int_a^b (w_{2i'} - w_{2\nu+2})^2 g dx = \int_a^b w_{2i'}^2 g dx - 2\int_a^b w_{2\nu} w_{2\nu+2} g dx + \int_a^b w_{2\nu+2}^2 g dx$$

approaches zero, as we have just seen. Hence

$$\int_a^b (W_2 - W_0)^2 g dx = 0,$$

and the continuous functions W_2 and W_0 are equal. We infer further that

$$\frac{d}{dx}(kW_0') + (\mu g - l)W_0 = W_1 g/c. \tag{12}$$

The norms of W_0 and W_1 are equal to c, so that either their sum or their difference is not identically zero. But $W_0 + W_1$ is a solution of (1) and (2) corresponding to $\lambda = \mu - 1/c$ and $W_0 - W_1$ is a solution corresponding to $\lambda = \mu + 1/c$.

Having one solution of the homogeneous problem, we need merely start with a function f orthogonal to it in order to arrive by the above process at a new one. In this way may be inferred the existence of an infinite set of solutions. A simple procedure would be to start with a function $f(x)$ having a break in its derivative but continuous. As such a function cannot be a finite sum of functions $c_i\varphi_i(x)$, it may, after φ_1, φ_2, φ_3, $\cdots$ have been found, be made successively orthogonal to these by subtraction of proper multiples of them, without reducing to zero, and so yield, in time, any function of the set.

The set containing all the solutions of (1) and (2) is then closed with respect to the class S. For, if not, there would be a function f of class S orthogonal to all φ_i, and starting the process developed above with this f, we should ultimately arrive at the contradiction that $W_0 + W_1$ or $W_0 - W_1$, that is, one of the φ_i itself, was orthogonal to the set.

CAMBRIDGE, MASS.,
August 23, 1920.

THE MATHEMATICAL WORK OF THOMAS JAN STIELTJES.

Œuvres complètes de Thomas Jan Stieltjes. Publiées par les soins de la Société Mathématique d'Amsterdam. P. Noordhoff, Groningen. Tome I, 1914: viii + 471 pp. Tome II, 1918: iv + 604 pp.

ONE can not assert that Thomas Jan Stieltjes was one of the great men of the earth. In fact he was not one of the greatest men in the narrower circle of his colleagues in mathematical investigation. But he was a man of fine talent who used the full strength of his powers in his researches; and in his short life (1856–1894) he did excellent work which deserves to be remembered. It is therefore fitting that his articles and memoirs should be brought together in the convenient form of a collected edition. In the two volumes of this work and in the two volumes which record the correspondence between Stieltjes and Hermite we have a complete record* of the scientific activity of Stieltjes and a clear and pleasing insight into his methods of work. The latter is more apparent in the letters, the less formal nature of which gave rise to a more intimate revelation of himself to his friend and later to the world.

His earlier work especially is marked by a careful and deep study of particular questions and the interest which he took in adapting algebraic and analytic formulas to numerical computation. This seems to have been due in considerable measure to the fact that he approached mathematics from the point of view of one engaged in astronomical work and only later gave up what was first conceived to be the object of his scientific life in favor of his studies in mathematics which during the years 1877 to 1883 gained a stronger and stronger grasp upon his thought.

Many of his discoveries were made empirically by incomplete induction from numerous examples developed from the beginning with the delight which he always evinced in numerical computation. In this respect his method has been compared

* See also the section devoted to the (unpublished) "method of Stieltjes" (pp. 357–362, 323) in Poincaré's memoir on the residues of double integrals in *Acta Mathematica*, volume 9, 1887.

with that of Gauss who is known to have discovered experimentally many of his beautiful theorems in the theory of numbers. This method has a peculiar power when it is employed by an intellect of sufficiently keen penetration to divine the general law in the midst of the special properties which belong to particular examples. The procedure of discovery in the case of Stieltjes seems to have been dominated by this empirical method. It seems to be true that nearly all the mathematical truths which he made known were discovered in this way before he was in possession of methods of proof; and that the latter were obtained afterwards by a penetrating analysis of the essential elements of his problem. The truths thus revealed by experiment were subjected to the acid test of logical demonstration and he was mostly free from the enunciation of results for which he had no adequate proof. His letters show the extreme care which he took in the matter of rigorous argumentation. His conversation is said to have shown a wide acquaintance with what may be called interesting mathematical phenomena which his patient calculations had brought to light but which he knew only partially through empirical evidence and not with the clarity and certainty and accuracy of demonstrated truth.

But he seems to have fallen at least once from the high plane of logical precision and accuracy which his published results usually occupied and to have stated a theorem for which he could give no satisfactory proof (though the theorem is probably true). He was engaged (Volume I, page 457) in an investigation of Riemann's celebrated conjecture concerning the distribution of the zeros of the Riemann zeta-function. He transformed the problem so that the truth of the theorem would follow from the convergence of a certain Dirichlet series whose coefficients depended on the function-values of a certain number-theoretic function $g(n)$. This function he examined for a certain property through the ranges of n from 1 to 1200, from 2000 to 2100, and from 6000 to 6100; and, finding the property maintained in these regions for n, he concluded that it was a universal property of $g(n)$. The proof by which he first satisfied himself logically of the existence of the property (see letter 79) he seems never to have published; and the inference to be drawn is that he later found it lacking in some point of accuracy or rigor.

One who reviews the work of Stieltjes at the present time

is spared the necessity of making an analysis of his separate memoirs, for this was done by E. Cosserat soon after the death of Stieltjes in a "Notice" of 62 pages appearing as the opening article of volume 9 (year 1895) of the *Toulouse Annales*. Here we have a résumé of each of the 84 papers now appearing in his Œuvres complètes, with the exception of the last three (which are minor contributions made up by his editors from his unpublished manuscripts). The student of the work of Stieltjes will find these abstracts very valuable for making a rapid survey of the extent of his contributions. [The forty-fifth abstract (and hence the forty-sixth) is perhaps misleading since it quotes the results of Stieltjes without reference to his failure to make good that one of his statements just mentioned in our preceding paragraph.] His great paper (number 80 of the Œuvres) on continued fractions is here discussed only briefly. But one interested in analyzing the contributions made by Stieltjes will certainly wish to examine this paper in full for himself, so that he will not suffer from the absence of a fuller review.

Again, a reviewer at the present time is under no obligation to give a sketch of the life of Stieltjes and relate his scientific investigations to what is thought of usually as the more human aspects of one's career, for this has already been done well by H. Bourget in his "Notice sur Stieltjes" in pages xi to xx of the first volume of the correspondence between Stieltjes and Hermite.

There remains then only the duty to call attention to the features of Stieltjes' work which are of outstanding interest or importance when viewed in the light of present knowledge or which are presumably of particular value to those who now carry the torch of science which he upheld so devotedly with all the strength of his fine talent.

For the purposes of discussion it is convenient to divide the active scientific life of Stieltjes into two parts. The first ends at the time when the influence of the Paris environment and labor and training began to be evident in his work, that is, about 1886 (though he went to Paris in April, 1885). The second period falls in the remaining eight years of his life. The physical bulk of his contributions, exclusive of the 113-page expository paper on the theory of numbers (which falls in the second period), was nearly equal in the two periods; but the two parts are of distinctly unequal merit. Whether

it was the intention of the editors to do so or not I do not know, but they have made the first and second volumes of his Œuvres cover each quite exactly one of these periods in his life, so that the physical division into volumes corresponds closely to a division of his work into distinct parts.

The two parts which may thus be distinguished and separated are yet intimately related and bear throughout the impress of their author's individuality. But the first is given more largely to special problems and shows more clearly the unfolding of the power of Stieltjes' thought, while the latter abounds more in the finished work of his maturer years. Yet in this first volume is to be found the beautiful contribution to the theory of cubic and biquadratic residues, his researches on mechanical quadratures, and his interesting paper on the variation of the density of the earth.

The last three or four years of the first period of his life were particularly full of activity. In this interval he was married, he began to develop what became a life-long intimate friendship with Hermite, and he had the stimulus due to his new abode in Paris. The change and development in the more external affairs of his life were repeated in, or at least had their counterpart in, the remarkable development of his spirit which took place at the same time.

The ingenious conceptions, the generating ideas, the germs of his later activity, multiplied during this interval and in the year or two which followed. To this epoch of his life nearly all his most interesting contributions are to be referred either for their completion or at least for their initial conception. In the few remaining years he developed and extended the ideas and solved some of the problems which had already arisen in his mind.

The centering of the more intensely creative activity of Stieltjes' life so largely in a single short period of it is perhaps instructive. In this period more than in any other new forces moved upon him from without and varied new delights (from new family ties, from the new friendship with Hermite, and from the new opportunity to give all his thought to mathematics) wrought upon his character and life and outlook to make him as it were a new individual. In this period the most of his essentially creative work was done. Is it perhaps true generally that the spirit of man yields its greatest return in new truth discovered when it is subjected to the exhilaration

of the maximum of pleasant change in environment and outlook and subject of interest? This question first arose in my mind and an affirmative answer pressed itself upon my thought when I was once studying the relation of the greater plays of Shakespeare to the character of his life at the time when they were produced; and I have often had occasion to observe a like correspondence in the external life and the more highly creative periods of numerous thinkers in widely separated fields of activity. The work of Stieltjes, while not that of a great master, seems to me nevertheless to be an instructive case in point. If my thesis is well founded it suggests a question of profound value as to how one shall maintain his own thought at the highest possible level of creative exaltation.

It remains to discuss briefly a few of the more important memoirs of the second period in Stieltjes' scientific life. The first of these is his Paris doctor's thesis on certain semi-convergent series. In his work as an astronomer he had often observed the usefulness of certain divergent series (analogous to the celebrated formula of Stirling in the theory of the gamma function) for the purposes of numerical computation and he realized that the theory of these series had never been put on a satisfactory basis. He set himself the task to make a systematic analysis of the matter so that at the end of his study he might be justified in looking upon the series as the asymptotic representation of one or more functions. He studied carefully certain divergent series of special importance and developed their properties so as to obtain from them the most exact information possible as to the numerical values of functions associated with them. He was proceeding largely by his empirical method of gradual approach to the central facts of importance and wide generality; and, if left to himself in this study, he would probably have pushed the investigation to a much wider range. But the genius of Poincaré had turned almost simultaneously and quite independently to this same problem which astronomy had dumbly set before the mathematician for so long a time, and he attacked the problem from a general point of view rather than through special cases; under the fire of his genius the leading secrets were brought to light. Doubtless much remains to be done with the general problem, but the work of Poincaré so overshadowed the simultaneous contribution of Stieltjes that the latter did not pursue the question in further extensive researches; and

his interesting and worthy contribution has largely fallen out of notice.

The most significant among the works of Stieltjes is his celebrated memoir entitled "Recherches sur les fractions continues" (printed on pages 402 to 566 of the second volume of the Œuvres). Here several partial investigations in his earlier work reach their full stature in completed theorems. The germs of some of the most fruitful ideas in the memoir go back as far as the later active years of the first period of his scientific life; and several partial results from his previous scientific activity are woven into it in their proper place and brought to a further stage of completeness than in the earlier papers. In volume 119 of the Paris *Comptes rendus* (at pages 630–632) one will find a report by Poincaré on this memoir. It is described as one of the most remarkable memoirs in analysis "written in the last years," and it is said to place its author in an eminent rank "in the Science of our epoch." In the "Notice" of Bourget we have an interesting, and even a touching, account of the way in which the last discoveries of Stieltjes, which made it possible for him to bring this memoir to completion, so fired the zeal of his spirit for several months as to keep his mind in the freshness of its power even though his last illness was already sapping his physical strength and bringing him to a state of weakness in which he was unable to take up any further labors. He died a few months after the completion of the memoir.

The reputation of Stieltjes can never go higher than the researches recorded in this memoir can take it; and it can never fall lower than the level to which this memoir would bring it; for, though several of his papers contain nothing more than the solutions of problems not inherently difficult, our judgment of the quality of his work will be based primarily on the character of his most worthy effort. In this memoir we find all the qualities of elegance and clarity and marked originality which are characteristic of his better work.

The continued fractions considered by Stieltjes are such that the incomplete quotients are alternately of the forms $a_{2n+1}z$ and a_{2n}, the a_i being real and positive, so that the fraction may be written in the form

$$\frac{1}{a_1 z}+\frac{1}{a_2}+\frac{1}{a_3 z}+\frac{1}{a_4}+\cdots.$$

His central result may be stated as follows: If the series Σa_n converges the fraction is oscillatory; the approximating reduced fractions of even order tend to one limit and those of odd order tend to another limit; in each of these two sequences of fractions the numerator and the denominator tend each to an integral function of "genre" zero all of whose roots are real and negative; the limiting form of each of these two sequences from the approximating reduced fractions is a function which is meromorphic throughout the finite plane and is decomposable into a series of simple partial fractions. If the series Σa_n diverges, the continued fraction is convergent and the limit is a function $F(x)$ which is holomorphic everywhere except (possibly) on the negative axis of reals; this function $F(x)$ can be represented by a certain definite integral which in certain cases may be replaced by a series of simple fractions. In connection with the proof of these results Stieltjes derives and utilizes (Chapter V) a remarkable theorem in the general theory of functions, treats (Chapter VI) his celebrated problem of moments and defines (Chapter VI) the now classic Stieltjes integral. An extension of this memoir of Stieltjes has been given by Van Vleck (*Transactions of the American Mathematical Society*, volume 4 (1903), pages 297–332). See also Van Vleck's Boston Colloquium lectures, pages 147–152.

Probably Stieltjes will be longest and most gratefully remembered for his introduction of the integral now called after his name. He employed it in 1894 incidentally to the solution of his continued fractions problem and did not undertake to develop its properties further than was needful for such a use of it. The great importance of the new limiting process was not at once realized and the possibility of its use remained latent for a number of years. But it is more recently coming into its own. Its place is made abundantly clear by Hildebrandt in this BULLETIN, volume 24, 1918, pages 178 ff., in his excellent statement of the reasons why it must be considered a conception of fundamental importance. Perhaps the two strongest are those which arise from the following two results: for every linear functional operation $U(f)$ on continuous functions $f(x)$ there exists a function $u(x)$ of bounded variation such that

$$U(f) = \int_a^b f(x)du(x),$$

the integral being taken in the sense of the definition of Stieltjes; a necessary and sufficient condition that every continuous function on (ab) may be uniformly approximated by linear combinations of a set of functions $[\varphi_1(x), \cdots, \varphi_n(x), \cdots]$ is that the only solution of the equations

$$\int_a^b \varphi_n(x)du(x) = 0, \qquad (n = 1, 2, \cdots)$$

for a $u(x)$ which for every x_0 shall satisfy the condition

$$u(x_0) = \tfrac{1}{2}[u(x_0 + 0) + u(x_0 - 0)]$$

is $u(x) =$ constant.

The way in which Stieltjes came to the introduction of his new limiting process is interesting as illustrating one extreme of the method of discovery. It is as far removed as possible from that method in which one sets out deliberately to extend or generalize conceptions previously found interesting or useful in order to ascertain what further things of value may be seen to grow out from them. Such a method a man of Stieltjes' scientific temperament could never have employed. His mind did not operate in the direction of extending known conceptions by meditating upon them; and apparently he could never have succeeded in working in this way, however well the method may be suited to a certain different and perhaps more robust scientific temperament. On the contrary he proceeded first with particular instances of the problem of a general investigation and solved a number of relatively simple special problems which arose in this way. Being cast down repeatedly by certain difficulties which he could not at first surmount and analyzing the tools which he had employed to ascertain why they did not carry him to the goal, he seems to have come to a realization that the integral which he had been using was not altogether as flexible as the exigencies of his problem demanded. Certain aspects of it he was able to look upon dynamically as a problem in moments. But the moment could not be expressed always in the form of an ordinary integral though it was certainly the limit (in a certain sense) of a sum much like that employed in the Cauchy-Riemann definition of integral. This limit afforded him his new integral. In this way arose one of the highly fruitful conceptions of recent mathematical analysis—one

which will probably play a role of fundamental importance in the further development of certain central branches of mathematics.

R. D. CARMICHAEL.

SHORTER NOTICES.

Opere di Evangelista Torricelli. Edited by GINO LORIA and GIUSSEPPE VASSURA. Faenza, 1919, 2 volumes. Volume I, part 1, xxxviii + 408 pp.; part 2, iv + 482 pp. + plates; Volume II, iv + 322 pp. + plates.

OF those who sat at the feet of Galileo (1564–1642) and from him received instruction and inspiration, two were permitted to enjoy this privilege only in the last weeks of his life. One of these, Viviani (1622–1703), was fifty-eight years his junior and was only twenty years of age when the great teacher passed away. Viviani survived Galileo by sixty-one years, the lives of the two bridging a span of nearly a century and a half. With propriety as well as with pride he could say, in his later life, that he was "postremus Galilei discipulus." In a way, however, Torricelli (1608–1647) could have said the same, for he too was one of the last of those who learned from the great master, although he died so early that he was not, like Viviani, the last disciple to pass away. Viviani signed his famous problem on the hemispherical dome by an anagram of the words "A postremo Galilei Discipulo," while Torricelli was proud to observe that the letters of his own name could be transposed to form the sentence "En virescit Galileus alter."

Of these two great disciples the more brilliant was Torricelli. With a span of life that was less than half as long as that of Viviani, he may be said to have accomplished twice as much, and the results of his labors have been set forth in the volumes under review.

Volume I, consisting of two parts, covers the work of Torricelli in the field of geometry and appears under the editorship of Professor Loria, while Volume II includes his academic lectures, his work in mechanics, and his writings in various minor lines, and is published "per cura" of Professor Vassura.

The introduction to Volume I gives a general survey of the life and works of Torricelli, contains much valuable bibliographical material, and includes considerable interesting information relative to the manuscripts which he left.

The inception of this edition dates from the action of the Congresso Internazionale di Scienze Storiche, held in Rome in 1903. At that time Professor Loria read a paper on "Un' impresa nazionale di universale interesse (pubblicazione delle Opere di Evangelista Torricelli)," with the result that the congress recommended that the Italian government assign to the R. Accademia dei Lincei the task of examining the manuscript works of Torricelli for the purpose of determining what ones should be printed, and also of deciding upon the works already published which should properly find place in a new edition. The recommendation was indorsed by other organizations, and on the occasion of the tercentenary of Torricelli's birth, in 1906, the Consiglio Comunale of Faenza determined to render financial assistance in the publication. The result of this decision was the publication of the present edition, some thirteen years later.

The editors have properly exercised their privilege of rearranging the material so as to present a unified appearance, transferring certain parts of the geometry, which appeared in 1644, to the second volume, and making other changes of a like nature as seemed necessary. This has resulted in a better sequence, and the only criticism that seems proper to make in this respect is that the text does not give, at the beginning of each of the separate divisions, a brief statement of the date and place of the first editions. Such information would be helpful to the bibliophile and historian and would not interfere in any way with the sequence chosen. Information of a somewhat similar nature is given with respect to the material published from hitherto unedited manuscripts.

In his more scientific works Torricelli wrote in Latin, but some of his popular essays and addresses are in Italian. In the present edition no attempt has been made at translation, and the several works appear as in the original. As to sequence, however, many changes have been made. Most of the Opera Geometrica (Florence, 1644) has been included in Volume I, which treats of geometry, but the portion "De motu gravium naturaliter descendentium, et proiectorum libri duo" has been transferred to Volume II, which relates in part

to mechanics. On the other hand, there has been added to the geometry considerable material from the "Discepoli di Galileo" in the manuscripts of the Collezione Galileiana at Florence (volumes 26, 27, 28, 29, 32, 33, and 37). This includes an appendix to lemma XX of the memoir "De dimensione parabolæ," and also a number of other chapters relating to geometry and now for the first time made generally accessible. Among these is Torricelli's Truffle Field ("Campo di tartufi"), with a number of interesting propositions relating chiefly to the circle; his notes "Contro gl'infinito" (mostly in Italian), evidently begun with the idea of placing a series of interesting paradoxes before his students; the essay on the center of gravity of sectors of a circle; the one "De maximis et minimis"; the "Nova per armillas stereometria," containing the particularly interesting chapter "De solidis vasiformis"; the essay "De infinitis spiralibus"; a brief treatment of conic sections; and his essay "De indivisibilibus," a chapter of particular interest to students of the period in which Cavalieri was beginning to pave the way for the calculus which developed towards the end of the seventeenth century. In this connection the reader will also wish to examine the second part of the essay "De centro gravitatis sectoris circuli," where the subject is treated "per geometriam indivisibilium."

The second volume contains the Lezioni Accademiche (in Italian) which were published in Florence in 1715, the essays on mechanics, the heretofore unpublished essay on Prospettiva Pratica (in Italian), and various miscellaneous letters and articles.

The reader who wishes to see Torricelli's classical discussion of the cycloid will find it on page 163 of Volume I, but should also consult the appendix, page 444.

Considering the work as a whole, the general reader will probably be surprised to find that Torricelli, who is generally looked upon as a physicist, wrote chiefly upon geometry. He will find, particularly in the hitherto unpublished chapters, considerable material that can be used with profit in connection with problem courses.

The matter here reprinted from the edition of 1644, judging from several random selections, shows a commendable degree of care on the part of both the editors and the publishers. As would naturally be expected in a work of this size, there are various typographical errors, as in the cases of Huygegs

for Huygens (I, 232), and Withe for White (ibid.). Exception may also be taken to the positive assertion that the birthplace of Thomas White was Hutton, which seems to be only a probability. There are also numerous misprints such as scguento for seguento (I, 294). Matters of this kind, occurring only casually, are too trivial to mention in detail in a review. The most serious defect in the work is the absence of an index, the tables of contents not being sufficiently complete to enable a reader to find easily the particular subject which he wishes to investigate, particularly in connection with the notes.

Aside from the introduction, the work of the editors consists chiefly in the arrangement of the material, with a few important notes such as the one by Professor Loria at the end of the chapter "De tactionibus" (Volume I, page 291). On the whole, the edition is a very satisfactory one, and it is another testimonial to the remarkable scientific and productive powers of Professor Loria.

DAVID EUGENE SMITH.

An Introduction to String Figures. By W. W. ROUSE BALL. Cambridge, W. Heffer and Sons, 1920. 38 pp.

IN the spring of 1920 Mr. Ball gave a lecture at the Royal Institution, London, on simple string figures and their history, and this lecture has now appeared in pamphlet form, designed to set forth, as the title page asserts, "an amusement for everybody." Much of the information given in the essay is already familiar to those who are acquainted (and what student of mathematics is not?) with Mr. Ball's Mathematical Recreations (fifth edition, chapter XVI, page 348), but there is a certain amount of added material in the present publication. On the other hand some of the figures mentioned in the Recreations are not given here. For those who do not have the larger work at hand, this pamphlet will be found of interest.

DAVID EUGENE SMITH.

Solutions of the Examples in a Treatise on Differential Equations. By A. R. FORSYTH. London, Macmillan and Company, 1918. 249 pages.

THIS volume should serve as a time-saver to those who are giving the usual course in differential equations. Since the solution of a differential equation so often depends upon selecting the proper ingenious device, even the experienced

mathematician may find the working out of a particular example to require in some cases a considerable amount of labor. Unless one has already completed a card catalogue or a note book containing his own solutions of such a wide range of examples as is to be found in Forsyth's Treatise, a work like the present will prove of great value.

The examples worked out in the German edition of the Treatise included only those contained in the first and second English editions. As is well known, the later editions contained a great many additional examples. The present volume includes the solution of these. All the examples have been worked out by Professor Forsyth himself, and, with possibly three exceptions, all were found to be solvable in the usual sense of the term.

CHARLES N. MOORE.

Space, Time and Gravitation; an Outline of the General Relativity Theory. By A. S. EDDINGTON. Cambridge, University Press, 1920. vii + 218 pp.

EDDINGTON has a pleasant style even when engaged in technical exposition. His Stellar Movements makes very interesting reading for any mathematician who likes to see what mathematics has recently done toward unraveling the structure of the sidereal universe. This style is a necessity when one tries to write a semi-popular account of Einstein's new theory. On the whole Eddington has succeeded in making the matter clear without appeal to too much mathematics. Of course the person who has absolutely no mathematical outlook beyond the high-school course will have difficulty in appreciating even the Prologue; but let us say a college graduate who has had his calculus, taught in no too formal fashion,—he will find the book possible. The physicist, the not too ignorant philosopher will welcome the chance to study the theory in its elemental simplicity. Einstein's own treatment and that of his followers is about as instructive to the beginner as lectures on generalized (Lagrangian) coordinates would be in collegiate physics as a first treatment of mechanics or Lamé's work as an introduction to the notion of potential.

Eddington begins with a Prologue on What is Geometry written in the form of a Platonic dialogue between an experimental physicist, a pure mathematician, and a relativist.

The three points of view are well done and spiritedly. He should have had a philosopher, but probably he shrank from abandoning his clear and definite style in an attempt to accord or even record such divergent points of view as are taken by Whitehead and Walker as to the philosophic significance of general relativity and whether idealism or realism prevails by virtue of it.*

A discussion of the FitzGerald contraction leads up to relativity (old style, 1905). Then the world of four dimensions (x, y, z, t) is dwelt upon, but no more than is necessary to the uninitiated. The fourth chapter discusses fields of force—the equivalence postulate. Then Kinds of Space serves as the title of a chapter which explains, as well as may be, the meat of non-euclidean geometry, the theory of curvature, the analysis of tensors. A hard chapter to write. (But we wonder whether it is true that Riemann, Christoffel, Ricci and Levi-Civita never dreamt of a physical application of their analysis (page 89). We seem to recall that Ricci, rather early, applied his absolute calculus to problems in elasticity, and Levi-Civita to problems in potential theory. Christoffel is les s known to us, but Riemann we always regarded as physicist quite as much as mathematician, and as a dreamer who dreamt many a relationship between the two.)

It may be well to ponder upon the statement (page 92): "I prefer to think of matter and energy, not as agents causing the degrees of curvature of the world, but as parts of our perceptions of the existence of the curvature." Relative to the older physics this is a turn-about quite Copernican. We have been in the habit of separating our difficulties—space, time, matter, energy, electricity, etc. The new style is to lump them all together and blame it on the way the world (four-dimensional) is made or on the way we make our measurements. It may be the better way. Only the future can tell. Certainly it does not seem the simplification that the Copernican theory represented as against the Ptolemaic. The great advances in science are usually ear-marked by simplicity,—the sort of thing the man in the street can understand if attentive. Einstein is reported to have alleged that not over a dozen in the world could read his book. That is not the way advances in science come,—but this may be

* See "Space, Time and Gravitation," by E. B. Wilson, *The Scientific Monthly*, March, 1920, especially p. 233.

the exception that proves the rule, or Einstein may merely be truthful and we may be fooling ourselves in thinking that the way of scientific advance is natural and easy, once it is pointed out by the pioneer.

Chapter VI is on the New Law of Gravitation and the Old Law. The defects of Newton's law are pointed out and the curvature of the space-time world in the neighborhood of matter is explained in detail with the aid of the differential

$$ds^2 = -dr^2/j - r^2d\theta^2 + jdt^2, \qquad (j = 1 - 2m/r),$$

of the manifold. The diagrams and analogies are suggestive and effective. Next comes a chapter on weighing light and among some excellent statements we find the puzzling one that: Possibly the tails of comets are a witness to the power of the momentum of sunlight which drives outwards the smaller or the more absorptive particles—puzzling until we surmise that probably reflection is not of much importance on account of low albedo. It is in this chapter that an account is given of the results of the eclipse observations at Sobral and Principe. Other tests of the theory form the subject of Chapter VIII,—perihelion of Mercury, minor corrections in the elements of other planetary orbits, shift of the spectral lines, and the philosophical test of the Principle of Equivalence as applied to the clock (page 131) This last argument is particularly interesting if somewhat dangerous, as the author seems to realize.

The chapter on momentum and energy brings in the ordinary relativist law for the change of mass with velocity But the chief interest lies in a quasi-philosophic, speculative attitude wherein gravitation and inertia are identified. If this has appeared in the writings of others, it has at least escaped our attention. There is (page 141) a suggestion that corresponding to any *absolute* property of a volume of a world of four dimensions (the property chosen by Einstein being curvature) there must be four *relative* properties which are conserved (in the Einstein theory, the conservation of energy and of momentum); and that this might be made the starting-point of a general inquiry into the necessary qualities of a permanent perceptional world. The inquiry should be made. Indeed, the statement would seem to place the Einstein theory of gravitation in the class of an astute guess for which such corroboration as the bending of light by definite quantitative

amount and the correct figure for the advance of the perihelion of Mercury could not be expected. It must be possible to modify the curvature by introducing other absolute properties in conjunction with it so as to leave the major fact of (Newtonian) gravitation unaltered but seriously to alter the minor corrections to it. Action, i.e., mass or energy multiplied by time, is given a fundamental position as density multiplied by a four dimensional volume, action as the curvature of the world, and reference is (later) made to the fundamental h of Planck's quantum theory. The curvature of the world in water (density 1 gm./cm.^{3}) is the same as that of space in the form of a sphere of 570,000,000 km. radius or, in time units, of radius about 1/2 hour. A homogeneous sphere of water of this radius would exhaust all space-time. This mass is small compared with many estimates of that of the sidereal universe. Apparently, then, that mass could never get together except in a very much rarer condition than water.

"Towards Infinity" is the title of Chapter X. Here is considered the contrast between translation and rotation. Can an observer accept as a possibility a geocentric system with non-rotating earth? He would have to have something to take care of the centrifugal force $\omega^2 r$. Would a distribution of gravitating masses accomplish this? The author appears to give it up, at least as a problem in physics and to believe there is a real distinction physically if not metaphysically between rotation and translation with respect to absoluteness. (Of course on the Newtonian theory a uniform distribution of negatively gravitating matter about any center will produce a repulsive force proportional to the distance.) A few words are given to acceleration. The curved spaces of de Sitter (spherical) and Einstein (cylindrical) are discussed. The author admits that things are getting speculative and less definite than before and that he is becoming bewildered. Almost all students of generalized relativity must at times share his sensations. One phrase we quote (page 165): The reader will see how our search for an absolute world has been guided by a recognition of the relativity of measurements of physics. This seems sound.

The penultimate chapter on electricity and gravitation gives a sketch of Weyl's theory, a new geometry in which a rod taken around a circuit in space and time changes length, thereby becoming responsible for the sensations of electric

shock. The theory is perplexing but it certainly has some points of philosophic advantage over Einstein's theory, besides leading to the electromagnetic equations. In it *action* is a pure number; there is so and so much *action* in a given region of space-time independent of the coordinates used or the unit of measures. The author thinks that in some way this pure number must be connected with *probability* so that the principle of least action becomes the principle of greatest probability. This is very attractive and, in the present state of our theory, very suggestive. There is a final chapter on the Nature of Things and an Appendix containing mathematical notes.

We agree with Eddington that H. Weyl's Raum, Zeit, Materie (recently appearing in a third edition) is the best treatise on the new relativity; his own is undoubtedly the best general presentation.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
August 15, 1920.

Researches in Physical Optics. Part II: Resonance Radiation and Resonance Spectra. By R. W. WOOD. New York, Columbia University Press, 1919. viii + 184 pp. + X pl.

THERE is no doubt that the theory of spectra of all sorts, including resonance spectra, offers opportunity for mathematical work, both on the side of the dynamics or kinematics of hypothetical atoms or molecules and on the side of empirical nomographic or curve-fitting studies of spectral series. Considerable has been accomplished in both directions but further studies will be necessary before anything approaching satisfaction relative to our knowledge of the intimate parts of optics and of the constitution of matter is reached. Moreover, unless some extraordinary genius like Willard Gibbs appears, to do for this field what he did for physical chemistry, much additional experimental knowledge must be acquired and digested. It is this experimental foundation, with respect to resonance spectra, which Professor Wood is developing and expounding in the researches recently appearing from the Columbia University Press. The book, as written, interests the experimental and descriptive physicist rather than the mathematician, even though he be a mathematical physicist.

E. B. WILSON.

NOTES.

THIS issue of the BULLETIN marks its official transfer to a new Committee of Publication. Since the material for the BULLETIN is prepared far in advance of publication, however, the actual transition will be gradual. Material for the February number, in fact, and some of the material for still later numbers, had been gathered and to some extent arranged by the preceding editors.

The new Committee of Publication, and their associated editors, enter upon their task with mixed feelings and with no better programme than to maintain the high record established in the past. To many it may seem surprising that no acknowledgment of the indebtedness of the Society to those who are now relinquishing control appears in the present number. This defect is explained essentially in the preceding paragraph.

The present editor and his associates bespeak for the BULLETIN under their management the cordial support of contributors and of members of the Society, and forbearance for faults that are inevitable in new and inexperienced direction.

THE seventy-third meeting of the American association for the advancement of science was held at Chicago, December 27 to January 1, under the presidency of Dr. L. O. HOWARD. Professor D. R. CURTISS was vice-president of section A (mathematics). On December 29, Professor O. D. KELLOGG delivered his address as retiring vice-president of the section on "A decade of American mathematics," before a joint session with the Mathematical Association of America and the Chicago section of the American Mathematical Society.

THE concluding (October) number of volume 21 of the *Transactions of the American Mathematical Society* contains the following papers: "Minima of functions of lines," by ELIZABETH LESTOURGEON; "Invariants of infinite groups in the plane," by E. F. SIMONDS; "On triply orthogonal congruences," by J. B. SHAW; "A set of properties characteristic of a class of congruences connected with the theory of functions," by E. J. WILCZYNSKI; "On the equilibrium of a fluid

mass at rest," by J. W. ALEXANDER; "Concerning approachability of simple closed and open curves," by J. R. KLINE.

THE following mathematical papers have appeared in recent volumes of the *Journal of the United States Artillery:* volume 50 (1919): "A method of computing differential corrections for a trajectory," by G. A. BLISS; volume 51: "Numerical integration of differential equations," by F. R. MOULTON; "Equations of differential variations in exterior ballistics," by W. E. MILNE; "The use of adjoint systems in the problem of differential corrections," by G. A. BLISS; "Rotating bands," by OSWALD VEBLEN and P. L. ALGER; "A method of computing differential corrections for a trajectory," by G. A. BLISS; volume 53 (1920): "On weighting factor curves in flat fire," by J. F. RITT; "Mirror and window position finders," by W. C. GRAUSTEIN.

RECENT numbers of the *Proceedings of the National Academy of Sciences* contain: volume 5, number 9 (September, 1919): "Radiation from a moving magneton," by H. BATEMAN; number 12 (December): "Conditions necessary and sufficient for the existence of a Stieltjes integral," by R. D. CARMICHAEL; "Transformations of cyclic systems of circles," by L. P. EISENHART; volume 6, number 1 (January, 1920): "The commutativity of one-parameter transformations in real variables," by A. C. LUNN; number 2 (February): "Groups generated by two operators, s_1, s_2, which satisfy the conditions $s_1^m = s_2^n$, $(s_1 s_2)^k = 1$, $s_1 s_2 = s_2 s_1$," by G. A. MILLER; number 3 (March): "A kinematical interpretation of electromagnetism," by LEIGH PAGE; "Note on geometrical products," by C. L. E. MOORE and H. B. PHILLIPS; number 4 (April): "The starting of a ship," by J. K. WHITTEMORE; "A thermodynamic study of electrolytic solutions," by F. L. HITCHCOCK; "Functionals invariant under one-parameter continuous groups of transformations in the space of continuous functions," by I. A. BARNETT; number 5 (May): "On Kummer's memoir of 1857 concerning Fermat's last theorem," by H. S. VANDIVER; number 6 (June): "On the distortion in conformal mapping when the second coefficient in the mapping function has an assigned value," by T. H. GRONWALL; "On the connection of the specific heats with the equation of state of a gas," by A. G. WEBSTER; "On the conformal mapping of a family of

conics on another," by T. H. GRONWALL; number 7 (July): "On the class number of the field $\Omega(e^{2i\pi/p^m})$ and the second case of Fermat's last theorem," by H. S. VANDIVER; number 9 (September): "On a differential equation occurring in Page's theory of electromagnetism," by H. BATEMAN; "A new proof of a theorem due to Schoenflies," by J. R. KLINE; "On the structure of finite continuous groups with exceptional transformations," by A. C. LUNN.

THE following doctorates in mathematics were conferred by the University of Pairs in 1920: B. GLOBA-MIKHAÏLENKO: "I. Contribution à l'étude des formes d'une masse fluide en équilibre. II. Méthode de Ritz pour l'équilibre, sous l'action d'une charge verticale donnée, d'une lame rectangulaire horizontale encastrée sur tout son contour,"; M. JANET: "I. Sur les systèmes d'équations aux dérivées partielles. II. Sur les congruences de droites."

IN acknowledging the gift to the American Mathematical Society's library of the thirteenth volume of the complete edition of Huygen's works, published by the Dutch Society of Sciences, it was referred to as the final volume of the set (see BULLETIN, volume 27, page 94). This is far from being correct, as several others are expected to appear; in fact, volume 14, published in 1920, has already reached us. This contains his work on the calculus of probabilities, "Van rekeningh in spelen van geluck," 1656–1657, with several appendixes of later date, and various papers on pure mathematics that appeared from 1655 to 1666.

THE Royal society of London has conferred a Royal medal on Professor G. H. HARDY, for his researches in pure mathematics, especially in the analytic theory of numbers.

PROFESSOR A. SOMMERFELD, of the University of Munich, has been awarded the Vahlbruch prize of the Academy of sciences of Göttingen. He has also been elected corresponding member of the Berlin Academy of sciences, and foreign member of the Royal Swedish academy at Upsala.

PROFESSORS A. EINSTEIN and M. PLANCK have been elected corresponding members of the Danish Society of sciences.

PROFESSOR H. WILSKI, of the technical school at Aix, has been elected a member of the Leopoldinisch-Carolinische Akademie at Halle, and Professor E. MÜLLER, of the Vienna technical school, an associate of the Austrian section of that academy.

THE philosophical faculty of the University of Giessen has conferred the honorary degree of doctor of philosophy on Professor C. KOSTKA, of Insterburg.

PROFESSOR E. COHN, formerly of the University of Strassburg, has been appointed honorary professor of theoretical physics at the University of Freiburg.

PROFESSOR C. CRANZ has been appointed professor of theoretical physics at the Charlottenburg technical school.

PROFESSOR E. FISCHER, of the University of Erlangen, has been appointed professor of mathematics at the University of Cologne.

AT the Dresden technical school, Professor G. KOWALEWSKI, of the German University at Prague, has been appointed professor of mathematics, Dr. M. LAGALLY, of Munich, professor of applied mathematics, and Dr. L. FÖPPL, of the University of Würzburg, professor of the theory of stability, hydrodynamics, and aerodynamics.

DR. R. GRAMMEL, of the University of Halle, has been appointed to a professorship at the Stuttgart technical school.

DR. W. LIETZMANN has been appointed professor of the teaching of the exact sciences at the University of Göttingen.

PROFESSOR H. LIEBMANN has been appointed professor of mathematics at the University of Heidelberg.

ASSOCIATE professor H. ROTHE, of the Vienna technical school, has been promoted to a full professorship of mathematics.

PROFESSOR E. STEINITZ, of the Breslau technical school, has been appointed professor of mathematics at the University of Kiel.

PROFESSOR R. WEITZENBÖCK, of the German technical school at Prague, has been appointed professor of mathematics at the technical school at Graz.

PROFESSOR A. FÖPPL, of the Munich technical school, has retired from active teaching.

THE following persons have been admitted as privatdocents: Dr. J. NIELSEN, for pure and applied mathematics, at the University of Hamburg; Dr. O. v. GRUBER, for applied mathematics, at the Munich technical school; Dr. G. WIARDA, for mathematics, at the University of Marburg.

PROFESSOR J. HADAMARD has been appointed professor of mathematical analysis at the Ecole centrale des arts et manufactures at Paris.

M. CERF has been appointed associate professor of mathematics at the University of Dijon.

DR. C. G. KNOTT has been appointed to the newly instituted office of reader in applied mathematics at the University of Edinburgh.

MR. F. J. HARLOW, head of the department of mathematics and physics at the Sir John Cass Technical Institute, Aldgate, has been appointed principal of the Municipal Technical College, Blackburn.

MISS NORA I. CALDERWOOD and Mr. T. A. LUMSDEN have been appointed assistant lecturers in mathematics at the University of Birmingham.

AT the University of Nebraska, Dr. T. A. PIERCE has been promoted to an assistant professorship of mathematics, Mr. W. M. BOND, Mr. O. C. COLLINS, of Oxford University, and Mr. C. R. SHERER have been appointed instructors, and Miss C. RUMMONS assistant instructor.

IN the department of mathematics at the University of Illinois, associate professor R. D. CARMICHAEL has been promoted to a full professorship; Dr. C. F. GREEN, Dr. L. L. STEIMLEY, and Dr. B. MARGARET TURNER have been appointed instructors; Professor F. R. SMITH, on leave of absence from Pennsylvania State College, has been appointed associate.

DR. LUDWIK SILBERSTEIN, author of works on mathematical physics, and formerly lecturer at the University of Rome, has accepted a research position with the Eastman Kodak Company at Rochester.

PROFESSOR HARRIET GLAZIER, on leave of absence from the Western College for Women, has been appointed professor of mathematics at the Southern Branch of the University of California for the current academic year, during the absence of Professor MYRTIE COLLIER.

AT the University of Pennsylvania, assistant professor F. H. SAFFORD has been promoted to a full professorship of mathematics, and Mr. H. M. GEHMAN and Mr. R. W. HARTLEY have been appointed instructors. Dr. R. A. ARMS has resigned, to accept a professorship at Pennsylvania College, Gettybsurg.

IN the department of mathematics at the United States Naval Academy, assistant professors J. A. BULLARD, A. DILLINGHAM, and J. N. GALLOWAY have been promoted to associate professorships, and Mr. M. A. EASON, Mr. H. H. GAVER, Mr. H. M. ROBERT, Jr., and Dr. W. F. SHENTON to assistant professorships; Mr. E. A. BAILEY, Mr. A. J. BARRETT, Mr. E. R. C. MILES, and Mr. A. A. ROBINSON have been appointed instructors.

MR. OSCAR SCHMIEDEL has been appointed professor of mathematics at Nebraska Wesleyan University.

DR. O. J. RAMLER has been promoted to an associate professorship of mathematics at the Catholic University of America.

PROFESSOR H. S. MYERS, of Huron College, has been appointed professor of mathematics at Southwestern College, Winfield, Kansas.

PROFESSOR R. A. WELLS of Park College, has been appointed associate professor of mathematics at the Michigan State Normal College.

MR. CORNELIUS GOUWENS, of the University of Kansas, has been appointed assistant professor of mathematics at Iowa State College.

PROFESSOR C. E. HORNE, of the University of Porto Rico, has been appointed dean of the college of agriculture and mechanic arts at the university of Mayagüez, P. R.

ASSISTANT professor F. C. KENT has been promoted to an associate professorship of mathematics at the Oregon Agricultural College.

MR. H. L. OLSON, of the University of Wisconsin, and Mr. H. A. SIMMONS have been appointed instructors in mathematics at the University of Michigan.

AT the University of Alberta, Mr. T. H. MILNE, of the University of Toronto, has been appointed lecturer, and Mrs. E. T. MITCHELL instructor in mathematics; Mr. GEORGE ROBINSON has resigned, to become assistant to Professor E. T. WHITTAKER in his mathematical laboratory at the University of Edinburgh.

MISS ANNA M. MULLIKIN and Miss HELMA HOLMES have been appointed instructors in mathematics at the University of Texas.

MR. C. E. HARRINGTON has been appointed instructor in mathematics at the University of Buffalo.

MR. L. M. GRAVES has been appointed instructor in mathematics at Washington University.

DR. A. R. WILLIAMS and Dr. C. D. SHANE have been appointed instructors in mathematics at the University of California.

PROFESSOR M. KRAUSE, of the Dresden technical school, died March 2, 1920, at the age of sixty-eight.

PROFESSOR L. POCHHAMMER, of the University of Kiel, died March 24, 1920, at the age of seventy-eight.

THE death is announced of the distinguished geometer Professor K. F. W. ROHN, of the University of Leipzig, on August 4, 1920, at the age of sixty-five.

THE death is announced of Professor E. SELLING, formerly of the University of Würzburg.

ASSISTANT professor T. R. DAVIES, of McGill University, died August 12, 1920, at the age of fifty-six.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALLY (E.). Géométrie synthétique des unicursales de troisième classe et de quatrième ordre. Paris, Gauthier-Villars, 1920. 6 + 100 pp.

BEUTEL (E.). Die Quadratur des Kreises. 2te Auflage. (Mathematisch-physikalische Bibliothek, Nr. 12.) Leipzig, Teubner, 1920. 57 pp.

CAJORI (F.). A history of the conceptions of limits and fluxions in Great Britain from Newton to Woodhouse. Chicago and London, Open Court, 1919. 300 pp. $2.00

CORPUT (J. G. VAN DER). Over roosterpunten in het platte vlak. (De beteekenis van de methoden van Voronoï en Pfeiffer.) (Akademisch Proefschrift.) Groningen, Noordhoff, 1919.

DICKSON (L. E.). History of the theory of numbers. Volume 2: Diophantine analysis. Washington, Carnegie Institution, 1920. 16 + 803 pp.

FEHR (H.). See LEVEUGLE (R.).

FUBINI (G.). Lezioni di analisi matematica. 3a edizione interamente rifusa. Torino, Società tipografico editrice nazionale, 1919. 8vo 8 + 512 pp.

GALE (A. S.) and WATKEYS (C. W.). Elementary functions and applications. New York, Holt, 1920. 20 + 436 pp. $2.60

LEBON (E.). Table de caractéristiques de base 30,030 donnant, en un seul coup d'œil, les facteurs premiers des nombres premiers avec 30,030 et inférieurs à 901,800,900. Tome I, 1er fascicule. Paris, Gauthier-Villars, 1920. 4to. 56 pp.

LEVEUGLE (R.). Précis de calcul géométrique. Avec une préface de H. Fehr. Paris, Gauthier-Villars, 1920. 8vo. 56 + 400 pp.

MACMAHON (P. A.). An introduction to combinatory analysis. Cambridge, University Press, 1920. 8vo. 6 + 71 pp. 7s. 6d.

MANSION (P.). Derniers mélanges mathématiques. Paris, Gauthier-Villars, 1920. 8vo. 4 + 188 pp. Fr. 10.00

ROUGIER (L.). Le philosophie géométrique d'Henri Poincaré. Paris, Alcan, 1920. 8vo. 208 pp. Fr. 9.00

VOOREN (W. L. VAN DER). Grenswaarden, eene inleiding tot de differentiaal- en integraalrekening. Groningen, Noordhoff, 1919. 8vo. 89 pp.

WATKEYS (C. W.). See GALE (A. S.).

WHITEHEAD (A. N.). The concept of nature. Tarner lectures delivered in Trinity College November 1919. Cambridge, University Press, 1920. 8vo. 10 + 202 pp.

II. ELEMENTARY MATHEMATICS.

BOREL (E.). Die Elemente der Mathematik. Vom Verfasser genehmigte deutsche Ausgabe besorgt von P. Stäckel. 2ter Band: Geometrie. 2te Auflage. Leipzig, Teubner, 1920. 16 + 380 pp.

CRANTZ (P.). Sphärische Trigonometrie zum Selbstunterricht. (Aus Natur und Geisteswelt, Nr. 605.) Leipzig, Teubner, 1920. 98 pp.

GRAY (J. C.). Number by development. Volume 3: Grammar grades. Philadelphia, Lippincott, 1919. 20 + 514 pp.

GUGLE (M.). Modern junior mathematics. Book 3. New York, Gregg, 1920. 13 + 246 pp. $1.00

HEDRICK (E. R.). Logarithmic and trigonometric tables. Revised edition. New York, Macmillan, 1920. 22 + 143 pp.

HJELMSLEV (J.). Elementær Geometri. 2den Bog. København, Gjellerup, 1919. 8vo. 111 pp.

KÖHLER (A.). Methodischer Führer und Ratgeber für den mathematischen Unterricht. Teil II: Planimetrie, Trigonometrie und Stereometrie. Stettin, Fischer und Schmidt, 1919. 395 pp.

NEUFELD (J. L.). Elementary algebra with a table of logarithms. Philadelphia, Blakiston, 1920. 12 + 383 pp.

STÄCKEL (P.). See BOREL (E.).

III. APPLIED MATHEMATICS.

ANDRIEU (—.). Les révélations du dessin et de la photographie à la guerre. Principes de métrographie. Paris, Gauthier-Villars, 1920. 8vo. 4 + 138 pp. Fr. 16.00

BAYLE (G.). Statique graphique. 2e édition. Paris, Librairie de l'Enseignement technique, 1920. 8vo. 164 pp. Fr. 15.00

BLOCH (W.). Einführung in die Relativitätstheorie. 2te, verbesserte Auflage. (Aus Natur und Geisteswelt, Nr. 618.) Leipzig, Teubner, 1920. 106 pp.

BUCHANAN (D.). See MOULTON (F. R.).

BUCK (T.). See MOULTON (F. R.).

CARL (A.). See KRAUSE (M.).

CHWOLSON (O. D.). Lehrbuch der Physik. 2ter Band, 1te Abteilung: Die Lehre vom Schall. Herausgegeben von G. Schmidt. 2te Auflage. Braunschweig, Vieweg, 1919.

DUBOSQUE (J.). Etudes théoriques et pratiques sur les murs de soutènement et les ponts et viaducs en maçonnerie. 6e édition. Paris et Liège, Béranger, 1920. 8vo. 380 pp. Fr. 48.00

EGERER (H.). Lehrbuch der technischen Mechanik in vorwiegend graphischer Behandlung. 1ter Band: Graphische Statik starrer Körper. Berlin, Springer, 1919. M. 20.00

FÖPPL (A.) und FÖPPL (L.). Drang und Zwang. Eine höhere Festigkeitslehre für Ingenieure. 1ter Band. München, Oldenburg, 1920.

FÖPPL (L.). See FÖPPL (A.).

FREUNDLICH (E.). Die Grundlagen der Einsteinschen Gravitationstheorie. 3te Auflage. Berlin, Springer, 1920. 96 pp. Geh. M. 6.80

GERDAY (C.). See MOULAN (P.).

GRIFFIN (F. L.). See MOULTON (F. R.).

HAHN (K.). Grundriss der Physik. Leipzig, Teubner, 1920. 8vo. 330 pp.

HUYGHENS (C.). Traité de la lumière. (Les Maîtres de la Pensée scientifique.) Paris, Gauthier-Villars, 1920. 18mo. 10 + 155 pp. Fr. 3.60

KRAUSE (M.). Analysis der ebenen Bewegung. Unter Mitwirkung von A. Carl. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1920.

LOEWY (A.). Mathematik des Geld- und Zahlungsverkehrs. Leipzig, Teubner, 1920. 8 + 273 pp.

LONGLEY (W. R.). See MOULTON (F. R.).

MACMILLAN (W. D.). See MOULTON (F. R.).

MARCELIN (A.). Propagation et réception des sons dans l'eau. (Thèse, Paris.) Paris, Imprimerie Lahure, 1920. 8vo. 92 pp.

MOULAN (P.). Cours de mécanique élémentaire à l'usage des écoles industrielles. 4e édition, revue et augmentée par C. Gerday. Paris et Liège, Béranger, 1920. 8vo. 1291 pp. Fr. 40.00

MOULTON (F. R.). Periodic orbits. By F. R. Moulton, in collaboration with D. Buchanan, T. Buck, F. L. Griffin, W. R. Longley, and W. D. MacMillan. (Carnegie Institution of Washington, Publication No. 161.) Washington, Carnegie Institution, 1920. 4to. 16 + 524 pp.

PIGEAUD (G.). Résistance des matériaux et élasticité. Paris, Gauthier-Villars, 1920. 8vo. 16 + 772 pp. Fr. 64.00

ROSE (W. N.). Mathematics for engineers. Part 2. London, Chapman and Hall, 1920. 8vo. 14 + 419 pp. 13s. 6d.

SCHMIDT (G.). See CHWOLSON (O.).

ADOLFO STAHL lectures in astronomy. Delivered at San Francisco by members of the staffs of the Lick Observatory, the Mount Wilson Solar Observatory, and the Students Observatory at Berkeley. Published by the Astronomical Society of the Pacific. Stanford University Press, 1919. $2.75

TRESLING (J.). Deformaties en trillingen in het vaste lichaam bij afwijkingen van de wet van Hooke, ook in verband met de toestandsvergelijking. Leyden, E. Ydo, 1919. 8vo. 5 + 83 pp.

VRIES (H. DE). Leerboek der beschrijvnde meetkunde. Eerste deel. Tweede druk. Delft, Technische Boekhandel en Drukkerij, J. Waltman, Jr., 1919.